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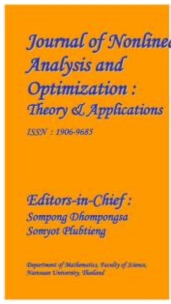

FRACTIONAL CALCULUS: RECENT DEVELOPMENT

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ABSTRACT
The concept of fractional calculus originated over 300 years ago, evolving through the revolutionary contributions of mathematicians. A multitude of mathematicians, from local to global arenas, have played pivotal roles in the evolution of fractional calculus, crafting innovative iterative and finite difference techniques to unravel linear and nonlinear fractional partial differential equations. Fractional calculus fills the voids that classical calculus has left behind and holds immense promise for future innovations spanning diverse domains. Over the last few decades, a vast array of applications and concrete instances of fractional calculus have emerged, culminating in the printing of upwards of 100 books and an incredible 100,000 scholarly papers concentrating on fractional differential equations. Fractional edge detection in the field of image processing, precise robot trajectory management in the field of robotics, fractional-order controllers within control theory, regular variation concepts in thermodynamics, fractional kinetics applications in the field of chemistry, fractional time evolution theories in the field of physics, discrete random walk frameworks for space-time fractional diffusion, cardiac tissue-electrode interactions in the field of biology, and the analysis of speech signals in the field of signal processing are all noteworthy breakthroughs. Additional applications include fractional-order speech modeling, fractional RC electrical circuits, modeling of children's physical development, fractional-order modeling of COVID-19, analysis of economic growth, applications in geo-hydrology, diffusion-wave equations for groundwater flow, Sturm-Liouville boundary value problems for integer-order systems, fractional mass-spring-damper systems, and fractional derivative models in biomedical and underwater sediment fields.

1. INTRODUCTION
The captivating field of fractional calculus has its beginnings in the significant date of September 30, 1695, when the distinguished mathematician Gottfried Wilhelm Leibniz introduced a compelling query regarding the characteristics of a derivative of non-integer order to L'Hôpital. This captivating domain of mathematics delves into the intricate properties and dynamic behaviors of derivatives and integrals that extend into the realm of non-integer (fractional) orders, equipping us with powerful tools for more comprehensive modeling across a multitude of disciplines [Kórus & Valdés, 2024; Kumar & Saxena, 2016].

Definition (1): Riemann- Liouville integral of order $\alpha > 0$ for a real valued function f(t) is

$$I^{\alpha}\mathrm{f}(t) = D_a^{-\alpha}\mathrm{f}(t) = \frac{1}{\Gamma(\alpha)}\int_a^t f(\tau)\,(t-\tau)^{\alpha-1}d\tau,\ t > a.$$

Definition (2): Riemann-Liouville fractional derivative of order $\alpha > 0$ for a real valued function f (t) is given as

$${}^{\mathrm{RL}}D_a^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, n-1<\alpha<n; n \in N \\ \frac{d^n}{dt^n}f(t),\ \alpha = n \in N \end{cases}$$

Definition (3): Caputo fractional derivative of order $\alpha > 0$ for a real valued function is

$$ {}^{C}D_{a}^{\alpha}f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}\int_{a}^{t}\frac{f^{n}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau, n-1<\alpha<n; n\in N \\ \frac{d^{n}}{dt^{n}}f(t),\ \alpha = n\ \in\ N \end{cases} $$

Definition (4): Mittag-Leffler function of one parmeter α is given as

$$E_{\alpha}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+1)}. \quad Re(\alpha > 0), z \in C.$$

A generalization of Mittag- Leffler function with two parameters α *and* β is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(\alpha k+\beta)}. \quad Re(\alpha > 0), z, \beta \in C$$

George Adomian (1988) unveiled the enchanting Adomian Decomposition Method (ADM), a gem that sparkled brightly from the 1970s through the 1990s. In Zhou (1986) brought forth the Differential Transform Method (DTM) specifically for electric circuit analysis, a versatile tool capable of unraveling both linear and nonlinear differential equations. Goyal and Mathur (2003) developed a theorem that combines the multidimensional Laplace transform, the multidimensional Varma transform, and the generalized Weyl fractional integral. This theorem has proven to be extremely useful in the field of evaluating the generalized Weyl fractional integrals of Fox's H-function in conjunction with the H-function of multiple complex variables. Saxena et al. (2005) made an attempt to estimate the Kober fractional fundamental integral operator for the foundational counterpart of the H-function. This endeavor contributed to the expansion of the q-fractional calculus theory. A pair of unified and extended fractional integral operators were investigated by Chaurasia and Gill (2011). These operators intimately connected the multivariable H-function, Fox's H-function, and a wide range of polynomials. Overall, the results of their investigation were quite impressive. In the paper that was published in 2009 by Mathai and colleagues, a comprehensive investigation into the qualities that are inherent to Mittag-Leffler functions and those of Mittag-Leffler type was presented. Yadav et al. (2010) developed two compelling hypergeometric operators of fractional q-integration, exhaustively deconstructing integration by parts as well as the connection theorem within the framework of the q-analogue of the Mellin Transform. Chaurasia and Gill (2011) were the ones that ignited the idea for an integral that elegantly combined the generalized Lauricella function with two H-functions of various complex variables. This integral was easily reducible to the F-function by means of creative parameter assignments. It was demonstrated by Chand (2012) that there are three novel theorems that entangle the I-function with a general class of polynomials. Through the application of strategic parameter adjustments, the core integral can be skillfully turned into Fox's H-function, the G-function, and the generalized Wright hypergeometric function. In their 2011 study, Chaurasia and Gill developed an equation that addresses the relationship between the Alpha-function and the internal blood pressure system. Kiran et al. (2010), along with other researchers hailing from the state of Maharashtra, have been actively engaged in the complex study of fractional calculus. The revolutionary contributions made by mathematicians who are passionate about their work have been the driving force behind the development of fractional calculus. In 1973, Truesdell presented the Gamma function, which was a fundamental component in the development of fractional calculus. Additionally, Truesdell revealed the integral by examining it through the lens of Euler's research, which was initiated in 1729.

$\Gamma(x) = \int_{0}^{\infty} e^{-t}\, t^{x-1}dt$, x > 0.

Lacroix (2021) developed $\frac{d^{m}x^{n}}{dx^{m}}$ = n(n-1)..............(n- m + 1)x^{n-m}, n , m ∈ N

$$= \frac{n!}{(n-m)!}\, x^{n-m};\ n \geq m,\ n, m \in N$$

$$= \frac{\Gamma(n+1)}{\Gamma(\,n-m+1)}\, x^{n-m},\ n > -1,\ n, m \in R$$

For particular when n = 1 and m = $\frac{1}{2}$, we get $\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$

Liouville's derivative: Liouville's first formula for fractional derivative is $D^{n}e^{ax}$, n ∈ ℕ for defining the fractional order α derivative of exponential function as $D^{\alpha}e^{ax} = a^{\alpha}e^{ax}$ where n ∈ ℝ. [Liouville's second formula for fractional derivative $D^{\alpha}x^{-\beta} = (-1)^{\alpha}\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\, x^{-\alpha-\beta},\ \beta \geq 0.$

Grünwald-Letnikov Fractional Derivative: Let f(t) ∈ C^{n}[a, b]. Then its first, second , third and nth derivatives are given as

$$f^1(t) = \frac{df}{dt} = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h}$$

$$\frac{d^2f}{dt^2} = \lim_{h\to 0}\frac{f'(t)-f'(t-h)}{h} = \lim_{h\to 0}\frac{f(t)-2f(t-h)+f(t-2h)}{h^2}$$

$$\frac{d^2f}{dt^2} = \lim_{h\to 0}\frac{f(t)-3f(t-h)+3f(t-2h)-f(t-3h)}{h^3}$$

And

$$\frac{d^nf}{dt^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}f(t-rh) = \lim_{h\to 0}\frac{\Delta_h^n f(t)}{h^n} \text{ ....(*)}$$

From (*), the Grunwald- Letnikov derivative is defined by

$$\frac{d^\alpha f}{dt^\alpha} = \lim_{h\to 0}\frac{\Delta_h^\alpha f(t)}{h^\alpha}, \alpha > 0.$$

Grünwald- Letnikov Fractional Derivative and Integral: For $\alpha \epsilon R$, Grünwald (1838-1920) and Letnikov (1837- 1888) defined the following

$$D_a^\alpha f(t) = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{r=0}^{\frac{x-a}{h}}(-1)^r\binom{\alpha}{r}f(t-rh), \text{ Where } \binom{\alpha}{r} = \frac{(-1)^r(-\alpha)_r}{r!} = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-r+1)\Gamma(r+1)}$$

The research conducted by Bhadane et al. (2011) in the Bulletin of the Marathwada Mathematical Society enriched the core principles of fractional calculus and unveiled its myriad applications, successfully deriving fractional integrals and derivatives for an array of functions. The powerful software Mathematica played a pivotal role in replicating these fractional integrals and derivatives. In another investigation by Kumar and Saxena (2016), mathematical modeling of diverse real-world systems in both engineering and science was explored, culminating in the formulation of differential equations. By employing fractional derivatives, the shortcomings of traditional integer-order models can be mitigated, offering a potent representation of the memory and hereditary characteristics inherent in processes. The manuscript also illustrated the superiority of fractional calculus over integer-order models through engaging examples like the mortgage dilemma and fractional oscillators.

The examination by Tarasov (2016) confirmed that the total fractional derivatives of smoothly varying functions are either classified as an integer-order derivative or a null operator. It was demonstrated that local fractional derivatives represent the limits of left-sided Caputo fractional derivatives, and the author further established that an intact Leibniz rule cannot be maintained for derivatives of order $\alpha \neq 1$.

A collection of books detailing the evolution of fractional calculus since 1985 encompasses:

(i) Denis Matignon, Gérard Montseny (Editors), "Fractional Differential Systems: Models, Methods, and Applications," ESAIM, vol. 5, 1998;

(ii) Journal of Vibration and Control, Special Issue: "Fractional Differentiation and its Applications," vol. 14, Sept. 2008;

(iii) Physica Scripta, "Fractional Differentiation and its Applications," T136, 2009;

(iv) Computers and Mathematics with Applications, Special Issue: "Advances in Fractional Differential Equations," vol. 59, Issue 3, Feb. 2010, pp. 1047-1376;

(v) "Fractional Differentiation and its Applications," vol. 59, Issue 5, March 2010.

Over the course of the last few decades, there has been a proliferation of novel applications and concrete examples of fractional calculus, as pointed out by Kumar and Saxena (2016). Remarkable applications can be found in the world of fractional calculus, which can be found in a variety of fields such as biology, mechanics, robotics, economics, control theory, chemistry, and the art of image processing. Remarkable advancements in various scientific fields showcase the power of fractional calculus and its applications. In image processing, the breakthrough technique of fractional edge detection, as presented by Mathieu et al. (2003) in "Fractional Differentiation for Edge Detection," has redefined how edges are identified with precision. In robotics, M.F. Silva, J.A.T. Machado, and A.M. Lopes (2003) introduced innovative approaches to trajectory control and analysis, enhancing robotic precision and movement. Fractional-order controllers have also become a cornerstone in control theory, explored in-depth by J.A.T. Machado (1997) and later by Y. Chen (2006), providing robust solutions for complex dynamic systems.

In thermodynamics, the concept of regular variation has been ingeniously applied to fractional damping elements, as discussed by Lion (1997) and further elaborated by Alexander in "On the Thermodynamics of Fractional Damping Elements." Similarly, in chemistry, V.U. Vladimir and S.

Renat (2013) examined fractional kinetics to model anomalous charge transport in semiconductors, dielectrics, and nanosystems, expanding the understanding of solid-state reactions in "Fractional Kinetics in Solids."

Physics has also embraced fractional calculus, with H. Nasrolahpour (2012) exploring both classical and quantum phenomena in "Time Fractional Formation," enriching the understanding of fractional time evolution. The application extends further into chemical physics, where Gorenflo, Rudolf, and colleagues (2002) developed discrete random walk models for space-time fractional diffusion, adding depth to diffusion studies.

In biology, Magin, R.L., and M. Ovadia (2006) delved into the complexities of the cardiac tissue-electrode interface, applying fractional calculus to better understand biological systems. Additionally, Assaleh, Khaled, and Wajdi M. Ahmad (2007) pioneered a novel approach to speech processing through "Modeling of Speech Signals Using Fractional Calculus," revolutionizing speech signal analysis.

Together, these advancements demonstrate the versatility and potential of fractional calculus, inspiring future researchers to further explore, extend, and generalize these findings across diverse disciplines. The Adomian Decomposition Method (ADM) and Differential Transform Method (DTM) have emerged as powerful, user-friendly tools for solving both linear and nonlinear ordinary and partial differential equations, whether deterministic or stochastic. These methods excel in delivering rapidly convergent solutions with high precision [Adomian, 1988]. Notably, ADM and DTM are also effective in addressing multi-pantograph equations, further expanding their range of applications.

The Variational Iteration Method (VIM), pioneered by Ji-Huan He, demonstrated impressive accuracy and efficiency in solving fractional differential equations through the work of Momani and Odibat (2008). Additionally, in 2007, they employed the Homotopy Perturbation Method (HPM) to solve fractional differential equations, establishing HPM as a viable alternative for tackling such mathematical challenges. Jafarian and Jafari (2013) adopted an innovative iterative technique developed by Daftardar-Gejji and Jafari (2005) to address both linear and nonlinear fractional diffusion and wave equations. Building on this, Daftardar-Gejji and Bhalekar (2010) further applied the method to fractional boundary value problems and evolution equations, demonstrating its versatility.

In numerical methods, Shen and Liu (2004) introduced an explicit finite difference approximation for the space-fractional diffusion equation, carefully analyzing its stability and convergence. Lin and Liu (2007) extended these studies by exploring the convergence and stability of high-order fractional methods, proposing an enhanced high-order approximation for fractional ordinary differential equations. These advancements illustrate the growing utility of analytical and numerical techniques in fractional calculus, inspiring further exploration in solving complex mathematical problems across various scientific domains.

**Fractional
Calculus in Robotics**:

Calculus serves as a vital instrument in articulating the dance of robot motion, encompassing strides both forward and backward, rotations, the orchestration of paths and trajectories, as well as the rhythm of velocity. It also enriches the realm of robot vision, influencing elements like the geometry of image formation, camera evaluations, and robotic command applications including force regulation and multivariable steering. Oustaloup (1991) unveiled a fractional-order controller and crafted the CRONE controller, ensuring a robust embrace of fractal intricacies. Podlubny (1999) introduced an extended concept of the PID controller, known as the P-Controller, in his work "Fractional-Order Systems and P-Controllers," published in IEEE Transactions on Automatic Control. This innovative controller incorporates an integrator of order λ and a differentiator of order μ, offering enhanced flexibility and precision in optimizing the dynamic behavior of control systems. The fractional-order nature of this controller enables finer adjustments, improving performance and robustness beyond what traditional integer-order controllers can achieve.

Fractional Calculus in Analytical Science:

Recently, the captivating field of fractional calculus has made strides in practical applications, including the detection of milk adulteration. Mabrook and Petty (2003) introduced a pioneering method for detecting water in full-fat milk using precise single-frequency admittance measurements. Similarly, Farag et al. (1983) devised an advanced technique to identify adulteration in cow and buffalo ghee with lard or margarine. Their approach involved a detailed analysis of the balance between saturated and unsaturated fatty acids, employing fractional crystallization and gas chromatography to ensure accurate detection. These studies exemplify the expanding influence of fractional calculus in addressing real-world challenges. Significant changes were observed in key fatty acids like 22:0, 18:1, 18:0, and 16:0. A regression equation was used to quantify the extent of adulteration, offering an effective approach for detecting lipid adulteration.

Milk adulteration poses serious concerns from both economic and public health perspectives. Mabrook and Petty (2003) explored electrical admittance spectroscopy to analyze fat and water composition in milk, as detailed in their work "Application of electrical admittance measurements to the quality control of milk" (Sensors and Actuators B: Chemical, 2002, pp. 136–141). Additionally, freezing point osmometry was utilized to measure water content in milk, as discussed by Büttel, Britta, Markus Fuchs, and Birger Holz (2008) in "Freezing point osmometry of milk to determine the additional water content – an issue in general quality control and German food regulation" (Chemistry Central Journal).

In earlier studies, Wolfschoun-Pombo, Alan Frederick, and Marco Antonio Moreira Furtado (1989) investigated milk adulteration by measuring casein-bound phosphorus and protein nitrogen content to detect whey in "Detection of adulteration of pasteurized milk with whey." Similarly, enzyme-based sensors were employed to detect urea contamination in milk, as highlighted by Renny, E.F. (2005) in "Enzyme-based sensor for detection of urea in milk" (Biotechnology & Biotechnological Equipment, pp. 198–201). However, these techniques were often complicated, time-consuming, and costly.

To address these limitations, Das, Siuli (2011) introduced an innovative constant phase angle-based impedance sensor capable of detecting adulterants such as water, liquid whey, and urea, as well as distinguishing between authentic and synthetic milk. This breakthrough is documented in "Performance study of a 'constant phase angle-based' impedance sensor to detect milk adulteration" (Sensors and Actuators A: Physical, pp. 273–278). Expanding on this work, Das further developed a fractional-order element-based impedance sensor that proved to be both robust and cost-effective for milk adulteration detection, as described in "Design and Development of Fractional Order Element Based Milk Adulteration Detection System" (Diss, 2012). These advancements underscore the growing relevance of innovative sensor technologies, particularly those grounded in fractional calculus, for effective and efficient milk quality control.

Fractional Differentiation based image processing:

In the field of image processing, fractional calculus has revolutionized methodologies like edge detection. The work "Fractional Differentiation for Edge Detection" (Signal Processing, 2003, pp. 2421-2432) by B. Mathieu, P. Melchior, A. Oustaloup, and Ch. Ceyral highlighted how fractional-order derivatives enhance the detection of fine edges while improving noise resistance. More recently, fractional calculus has advanced texture segmentation, enabling the detection of subtle objects and enhancing image quality. Notable contributions include Jia Huading and Pu Yifei's "Fractional Calculus Method for Increasing Digital Image of Bank Slip" (Proceedings of the 2008 Congress on Image and Signal Processing, vol. 3, pp. 326-330) and "Fractional Differential Approach to Detecting Textural Features of Digital Image and Its Fractional Differential Filter Implementation" (Sci. China Ser. F Inf. Sci., 51(9), pp. 1319-1339, 2008).

Fractional calculus has also made a significant impact on medical imaging. Hamid A. Jalap and Rabha W. Ibrahim, in their work "Texture Enhancement for Medical Images Based on Fractional Differential Masks" (Discrete Dynamics in Nature and Society, vol. 2013), demonstrated that fractional differential operators effectively extract subtle and indirect information, leading to improved image clarity. J.M. Blackledge further showcased the power of fractional calculus in "Diffusion and Fractional Diffusion-Based Image Processing" (EG UK Theory and Practice of Computer Graphics, Cardiff, 2009, pp. 233-240), applying the fractional Fourier Transform to suppress incoherent scattered light from random media, resulting in clearer astronomical images. Traditional edge detection methods rely on integer-order differentiation, with the gradient operator using order 1 and the Laplacian operator using order 2 [Kumar & Saxena, 2016]. However, the flexibility and precision offered by fractional-order differentiation provide a more nuanced approach, demonstrating the potential of fractional calculus to surpass conventional techniques in image processing.

Fractional calculus in Bio sciences:
O. S. Iyola and F. D. Zaman, in "A Fractional Diffusion Equation Model for Cancer Tumor," AIP Adv. 4, 2014, delved into the intricate world of cancer tumor cells by employing a fractional diffusion model that meticulously considers both the structural nuances and temporal fluctuations of tumor cell concentration alongside the lethality rate. They unveiled approximate analytical solutions through the q-homotopy analysis method (q-HAM), underscoring the critical need for a fractional-order perspective. In "On Fractional Order Cancer Model," Journal of Fractional Calculus and Applied Analysis, 3(2), 2012, E. Ahmed, A. Hashish, and F. A. Rihan explored the dynamic interplay between two immune effectors and cancer cells via a fractional-order framework, discovering that fractional-order systems excel in encapsulating tumor-immune interactions better than their integer-order siblings.
In "Fractional Calculus Applied to Model Arterial Viscoelasticity," Latin Am. Appl. Res. 38, 141-145, 2008, D. O. Craiem, F.J. Rojo, J.M. Atienza, G.V. Guinea, and R.L. Armentano ventured into the realm of a fractional model to investigate arterial viscoelasticity. The authors ingeniously adapted a conventional linear solid model, swapping out a dashpot for a spring-pot of order α. This fractional model adeptly forecasted relaxation responses within the span of one hour, boasting least square errors falling below 1%. The customized parameters facilitated the prediction of frequency responses akin to the documented complex elastic moduli of arteries. Their findings suggest that fractional models hold remarkable promise as innovative alternatives in the modeling of arterial viscoelasticity.
Richard L. Magin, in his work "Fractional Calculus Models of Complex Dynamics in Biological Tissues" (Computers and Mathematics with Applications, vol. 59, Issue 5, 2010), championed the use of fractional calculus to model the intricate behaviors of cells and tissues. This approach offers deeper insights into the complexity of molecular interactions and membrane dynamics, providing a more nuanced understanding of biological functions and the behavior of living systems. By capturing these complex processes, fractional calculus enhances the ability to analyze and predict biological phenomena with greater precision. According to Prakash (2024), several distinguished Indian researchers are making significant strides in the field of fractional calculus:

- Prof. H.M. Srivastava, University of Victoria, Canada
- Prof. K. Balachandran, Bharathiar University
- Prof. Varsha Gejji, University of Pune
- Prof. Subir Das, IIT (BHU)
- Prof. Loknath Debnath, University of Texas-Pan American, USA
- Prof. Lakshmikantham (deceased), Florida Institute of Technology, USA
- Prof. Mathai, McGill University, Canada & CMSS, Kerala
- Prof. Saxena, Jai Narain Vyas University, Jodhpur
- Prof. Anindya Chatterjee, Indian Institute of Science, Bangalore
- Prof. P.S.V. Nataraj, IIT Bombay
- Prof. S. Sen, IIT Kharagpur
- Prof. Gangal, University of Pune
- Mr. Shantanu Das, Scientist, Bhabha Atomic Research Centre, Mumbai
- Prof. Arijit Biswas, Jadavpur University, Kolkata
- Prof. R. Sahadevan, RIASM, University of Madras

The inaugural global symposium titled "Fractional Calculus and Its Applications" was orchestrated by B. Ross at the University of New Haven, USA, in June 1974, and garnered enthusiastic acclaim from the research community. Since 2004, the International Federation of Automatic Control (IFAC) has convened a distinctive international conference, "Fractional Differentiation and Its Applications," every two years.
Currently, there are four dedicated international journals and over 100 books published in the field of fractional calculus, with more than 100,000 research papers on fractional differential equations.
Developments in recent years:
Machado et al. (1997) compiled twenty special issues focusing on fractional calculus. Notably, in 2011 and 2012, approximately seven special issues were published on this topic, with a significant

increase in the number of publications from 2011 to 2014 compared to the period from 1999 to 2010. In addition, an impressive twenty unique special issues emerged during the years 2013 and 2014, showcasing the escalating enthusiasm and joint endeavors within the realm of fractional calculus. The essence of a fractional derivative intrinsically encompasses integration, and since it is defined over a span, it signifies a non-local operator. Consequently, fractional derivatives embody non-local operators as well. Fractional calculus broadens the horizons of classical calculus, transforming it into a powerful instrument for modeling and dissecting intricate phenomena across a myriad of domains, such as physics, biology, engineering, and economics (Kórus and Valdés, 2024).

Overview of published papers:

Article (1): "Imaga, Samuel A. Iyase, and Peter O. Ogunniyi" conducted an in-depth study on the existence of solutions for a mixed fractional-order boundary value problem at resonance along the half-line, integrating both Caputo and Riemann-Liouville fractional derivatives. Utilizing Mawhin's coincidence degree theory, they identified the necessary conditions for the existence of solutions, particularly when the dimension of the kernel of the linear fractional differential operator is two. Their work provides valuable insights into the mathematical foundations of fractional differential equations, extending the understanding of boundary value problems in fractional calculus.

Article (2): "Sheza M. El-Deeb and Luminita-Ioana Cotârla" delved into the intriguing characteristics of innovative subclasses of meromorphic p-valent functions within the realm of the punctured open unit disk. Through the clever fusion of q-derivative techniques and convolution methods, they unveiled a groundbreaking linear differential operator, meticulously exploring aspects such as distortion bounds, coefficient estimation, and the nature of convex families.

Article (3): "Ayub Samadi, Sotiris K. Ntouyas, Bashir Ahmad, and Jessada Tariboon" delved into an intricate nonlinear, non-local, and completely interconnected boundary value conundrum featuring generalized Hilfer fractional integral operators. They reimagined the challenge as a fixed-point dilemma by employing the fixed-point theorem, thereby demonstrating results of existence and uniqueness backed by three illustrative examples.

Article (4): "Ahmed Salem and Kholoud N. Alharbi" delved into the realm of moderate controllability solutions through two innovative methodologies: compactness technology inspired by Krasnoselskii's theorem and non-compactness validated via the Kuratowski measure of non-compactness alongside the Sadovskii fixed-point theorem. Additionally, they investigated a system with infinite delays characterized by Caputo fractional evolution equations, bolstered by illustrative numerical examples.

Article (5): "Isa A. Baba, Usa W. Humphries, Fathalla A. Rihan, and Juan E. Napoles Valdes" crafted an innovative fractional-order model for COVID-19, weaving in the complexities of indirect transmission, featuring a tapestry of six compartments in the Caputo framework. They revealed that the uninfected individuals could fall prey to the virus via both direct and indirect avenues of transmission. The research also embraced an analysis of optimal control and showcased numerical variants.

Article (6): "Constantin Fetecău and Costica Moroşanu" engaged in an enlightening dialogue about two pivotal subjects: a meticulous qualitative exploration of a diffusion dilemma characterized by nonlinear diffusion and cubic-type dynamic boundary conditions, where they elaborated on established findings through innovative mathematical frameworks aimed at depicting intricate physical phenomena; and the creation of a recursive fractional step-type method designed for approximating the nonlinear second-order reaction-diffusion challenge, coupled with the validation of convergence, error analysis, and the formulation of a conceptual numerical algorithm.

Article (7): "Bahtiyar Bayraktar, Péter Kórus, and Juan Eduardo Nápoles Valdés" delved into the dominion of convex functions, explored the broader category of general convex functions, and scrutinized differentiable functions characterized by their convex derivatives in absolute value. By leveraging the Jensen-Mercer inequality alongside its various adaptations for general convex functions, they ingeniously formulated Hermite-Hadamard-type fractional inequalities through the lens of non-conformable fractional integrals.

Article (8): In a fascinating study, Mohammad Faisal Khan, Suha B. Al-Shaikh, Ahmad A. Abubaker, and Khaled Matarneh introduced a differintegral operator specifically designed for m-fold symmetric functions, paving the way for a new class of close-to-convex bi-univalent functions. Their approach leveraged the Faber polynomial expansion technique, enabling deeper insights into the structure of these functions. Additionally, they explored the approximation of key mathematical bounds, including those of the general Taylor-Maclaurin coefficients, the initial coefficients, and the intriguing Fekete-Szegö functional, contributing valuable advancements to the field of complex analysis.

Article (9): "Asfond Fahad, Saad Ihsaan Butt, Bahtiyar Bayraktar, Mehran Anwar, and Yuan-heng Wang" have ingeniously crafted a novel fractional Bullen-type identity tailored for functions that are twice-differentiable, utilizing fractional integral operators. They unveiled a set of generalized Bullen-type inequalities, enriched with vivid graphical illustrations and applications that intertwine with modified Bessel functions, quadrature rules, and digamma functions. They demonstrated that enhanced Holder and power mean inequalities deliver superior outcomes for the upper limit, surpassing the results of classical inequalities.

Article (10): A remarkable integral identity was introduced for functions that are twice-differentiable, offering new insights into the behavior of convex functions. This identity paved the way for the formulation of innovative Hermite-Hadamard-Mercer-style inequalities, specifically designed for twice-differentiable convex functions. These inequalities serve as significant generalizations of several classical inequalities, such as the Hermite-Hadamard inequality, broadening their scope and relevance. Moreover, these newly established inequalities are not only theoretical enhancements but also practical tools, extending their applicability to various fields where convexity plays a crucial role, such as optimization, economics, and mathematical analysis. The versatility of these inequalities was showcased through illustrative applications, revealing their effectiveness in capturing more intricate properties of convex functions. This development emphasizes the role of the new inequalities as comprehensive extensions, enriching the mathematical framework by offering deeper insights and solutions for problems involving convexity and integrals.

According to the research conducted by Prakash (2024), the realm of fractional calculus has flourished due to the groundbreaking contributions of luminaries such as Leibniz, Bernoulli, Euler, Lagrange, Abel, Riemann, and a myriad of others. Leonard Euler (1729) introduced the Gamma function which plays important role in the development of Fractional Calculus and discovered the integral

$\Gamma(x) = \int_0^{\infty} e^{-t}\, t^{x-1} dt,\ x > 0.$

Lacroix (2021) developed $\frac{d^m x^n}{dx^m} = n(n-1)\ldots\ldots\ldots\ldots(n- m + 1)x^{n-m}$, n , m ∈ N

$$= \frac{n!}{(n-m)!}\, x^{n-m};\ n \geq m,\ n, m \in N$$

$$= \frac{\Gamma(n+1)}{\Gamma(\,n-m+1)}\, x^{n-m},\ n > -1,\ n, m \in R$$

For particular when n = 1 and m = $\frac{1}{2}$, it is obtained as $\frac{d^{\frac{1}{2}}x}{dx^{\frac{1}{2}}} = \frac{2\sqrt{x}}{\sqrt{\pi}}$

Liouville's derivative: Liouville's first formula for fractional derivative is $D^n e^{ax}$, n $\in \mathbb{N}$ for defining the fractional order α derivative of exponential function as $D^{\alpha}e^{ax} = a^{\alpha}e^{ax}$ where n $\in \mathbb{R}$.

Lowville's second formula for fractional derivative $D^{\alpha}x^{-\beta} = (-1)^{\alpha}\frac{\Gamma(\alpha+\beta)}{\Gamma(\beta)}\, x^{-\alpha-\beta}, \beta \geq 0.$

Grünwald-Letnikov Fractional Derivative:

Let f(t) ∈ C^n[a, b]. Then its first, second , third and nth derivatives are given as

$$f^1(t) = \frac{df}{dt} = \lim_{h\to 0}\frac{f(t)-f(t-h)}{h}$$

$$\frac{d^2f}{dt^2} = \lim_{h\to 0}\frac{f'(t)-f'(t-h)}{h} = \lim_{h\to 0}\frac{f(t)-2f(t-h)+f(t-2h)}{h^2}$$

$$\frac{d^2f}{dt^2} = \lim_{h\to 0}\frac{f(t)-3f(t-h)+3f(t-2h)-f(t-3h)}{h^3}$$

And

$$\frac{d^nf}{dt^n} = \lim_{h\to 0}\frac{1}{h^n}\sum_{r=0}^{n}(-1)^r\binom{n}{r}f(t-rh) = \lim_{h\to 0}\frac{\Delta_h^n f(t)}{h^n}\ \ldots\ldots\ldots\ldots(*)$$

From (*), the Grunwald- Letnikov derivative is defined by

$$\frac{d^{\alpha}f}{dt^{\alpha}} = \lim_{h\to 0}\frac{\Delta_h^{\alpha} f(t)}{h^{\alpha}}\ ,\ \alpha > 0$$

Grünwald- Letnikov Fractional Derivative and Integral: For $\alpha \in R$, Grünwald (1838-1920) and

Letnikov (1837- 1888) defined the following

$D_a^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{\frac{x-a}{h}} (-1)^r \binom{\alpha}{r}$f(t- rh), Where $\binom{\alpha}{r} = \frac{(-1)^r(-\alpha)_r}{r!} = \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-r+1)\Gamma(r+1)}$

The **fractional derivative** is inherently tied to the concept of integration, making it a **non-local operator**. Since integration spans an interval, the fractional derivative at a specific point depends on the entire range of function values over that interval.

- **Time-fractional derivatives** require all prior function values from t=0t = 0t=0 to t=t1t = t_1t=t1 to compute the derivative at t1t_1t1.
- **Space-fractional derivatives** rely on non-local values across a spatial domain, making them crucial in systems with spatially distributed parameters.
- Fractional derivatives are particularly effective in **modeling memory effects**, where the current state depends on past states, and they are widely used in **systems with distributed parameters** for capturing complex behaviors.

In recent decades, interest in the **stability, oscillation, asymptotic behavior, and existence and uniqueness of solutions** in fractional differential equations (FDEs) has grown significantly [Sharma, 2017]. The **non-local nature** and **weakly singular kernels** of fractional derivatives introduce analytical complexities that are absent in classical differential equations, often requiring numerical methods to obtain approximate solutions.

This edition weaves together diverse research, focusing on the **existence, uniqueness, multiplicity of solutions, solvability, stability, and oscillatory behavior** of fractional differential and discrete fractional equations. The **existence, uniqueness, and multiplicity of solutions** form the foundation of FDE theory. Notably, Cui and Zou (2013) explored **extremal solutions for nonlinear fractional differential systems** involving coupled four-point boundary value problems, employing **monotone iterative techniques** alongside the **method of upper and lower solutions**.

Yang (2019) tackled **second-order three-point boundary value problems with impulses**, applying **variational techniques** and **critical point theory** to obtain solutions. Similarly, Sun and Han (2023) examined **Sturm-Liouville boundary value problems**, providing **lower bounds for eigenvalues** and establishing constraints between real and imaginary parts.

Ding et al. (2017) developed high-order numerical algorithms for **Riemann-Liouville derivatives**, achieving accurate approximations of the **Riesz fractional derivative**. Using these methods, they created a **numerical framework for the Riesz fractional diffusion equation**, employing a **compact difference scheme** for first-order time derivatives.

Liu et al. (2014) introduced a **Dadras system** with complex variables, uncovering mesmerizing **four-wing hyperchaotic and chaotic attractors**. Their numerical experiments illuminated dynamic properties such as **Lyapunov exponents**, **fractal dimensions**, and **Poincaré maps**, showcasing the versatility of fractional systems.

Lv (2014) addressed the **existence of solutions for discrete three-point boundary value problems at resonance**, applying the **coincidence degree continuation theorem** with Riemann-Liouville fractional differences. Cheng et al. (2014) investigated **weakly singular discrete nonlinear inequalities** for **Volterra-type difference equations**, analyzing the uniqueness and behavior of solutions governed by **weakly singular kernels**.

Zhang et al. (2019) focused on **delay-dependent stability** in neutral systems with **mixed delays and nonlinear distortions**, employing **linear matrix inequalities (LMIs)** and a **Lyapunov functional approach** to derive stability criteria. Xiang et al. (2014) proposed new **oscillation criteria** for fractional differential equations with Liouville derivatives, utilizing **generalized Riccati functions** and inequality methods to analyze rhythmic behavior. Similarly, Zhang and Liu (2014) studied **oscillation conditions for half-linear neutral delay dynamic equations** on time scales using **Riccati transformations**.

To derive **closed-form transient solutions** for FDEs, researchers employed **circulant-type matrices** to extract **Routh-Hurwitz stability conditions** from eigenvalues of circulant matrices. Y. Gong et al. examined **circulant, left-circulant, and g-circulant matrices**, exploring their invertibility through **Jacobsthal and Jacobsthal-Lucas numbers** and presenting their determinants and inverses.

Baleanu and Agarwal (2014) applied **generalized fractional integral operators** involving **Appell's function**, formulating an **image equation for the generalized Gauss hypergeometric function**. These results laid the groundwork for **composition formulas** that integrate various fractional integral

operators with generalized Gauss hypergeometric functions, extending the theoretical landscape of fractional calculus.
This body of work demonstrates the growing influence of fractional calculus across mathematics and applied sciences, offering innovative solutions to complex problems while inspiring further exploration into uncharted areas of differential equations and dynamical systems.
Remarkable advancements in fractional derivatives and integrals have been achieved by a multitude of scholars, including "Holmgren, Davis, Hardy and Littlewood, Burkill, Copson, Zygmund, Bassam, Kuttner, Kesarwani, Kalisch, O'Neil, Kober, Gaer and Rubel, Love, Prabhakar, Rubin, Adams, Koeller, Ahern and Jevti, Glöckle and Nonnenmacher, Bagley and Torvik, Koeller, Ross, Srivastava, Saigo and Owa, Debnath, Agarwal, Lakshmikantham and Nieto, Magin, Engheia, Kilicman and Al Zhour, Machado, Kiryakova, Mainardi, and Gorenflo (2013)".
Numerous tomes on fractional calculus have been penned by "Oldham and Spanier, Podlubny, Magin, Kilbas, Srivastava and Trujillo, Sabatier, Agrawal and Machado, Diethelm, Monje et al., Tarasov, Ortigueira, Baleanu, Yang, Machado and Luo, Anastassiou, Klafter, Lim and Metzler, Meerschaert and Sikorskii, Baleanu, Diethelm, Scalas and Trujillo, Li and Zeng, Malinowska and Torres, Abbas, Benchora and N'Guérékata, Uchaikin, Richard, Carpinteri and Mainardi, Atanackovic, Pilipovic, Stankovic and Zorica, Yong, Jinrong and Lu, Guo, Xueke and Fenghui, Lorenzo and Hartley".
In 1884, Sonine crafted a solution for the enigmatic Abel-type equation and unveiled the Sonine condition, marking a pivotal moment in the realm of generalized fractional calculus. The second-kind Abel-type integral equation was introduced by Hille and Tamarkin in 1930, who innovated a generalized fractional derivative through the application of the "Mittag-Leffler function. Moreover, Erdélyi and Kober brought forth generalized fractional derivatives and integrals, skillfully employing the Euler gamma function.
Euler gamma function $\Gamma(x) = \int_0^\infty e^{-t}\, t^{x-1}dt$, $Re(x) \geq -1$ was given by Euler in 1729 which was proposed by Legendre in 1809 as
$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\,\Gamma(2x)\Gamma(x+\frac{1}{2})$, $2x \neq -1,-2,\ldots$

The Mittag- Leffler function:
Swedish mathematician Gosta Mittag- Leffler in 1903 proposed the Mittag- Leffler function as :
$$E_v(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(nv+1)}.$$
Hilfer (2000) introduced(proposed) the fractional derivative are the Riemann-Liouville fractional integrals.
$D^{\alpha,\beta}f(x) = I_0^{\beta(1-\alpha)}.\, D^{(1)}.I_0^{(1-\beta)(1-\alpha)}f(x)$ where $D^{(1)}f(x) = \frac{df(x)}{dx}$, $I_0^{\beta(1-\alpha)}$ and $I_0^{(1-\beta)(1-\alpha)}$
Gajda and Mgdziarz (2010) presented the fractional derivative in the form
$$D_+^{(\alpha)}f(x) = \frac{d}{dx}\int_0^x M(x-t)f(t)dt,$$
where the Laplace transform M(s) of the function $M(x) = \frac{1}{(s+\lambda)^\alpha + \lambda^\alpha}$
In 2015. Caputo and Fabrizio initiated the fractional derivative with exponential function as
$$D_x^{(\alpha)}f(x) = \frac{(2-\alpha)\Im(\alpha)}{2(1-\alpha)}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)\frac{df(t)}{dt}\,dt,$$
where $\Im(\alpha)$ is a parameter.
Zayernouri et al. (2015) suggested the fractional derivatives in the form
$$D_+^{(\alpha)}f(x) = \frac{e^{\lambda x}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{f(t)}{(x-t)^{\alpha+1}}e^{-\lambda t}dt \text{ and}$$
$$D_-^{(\alpha)}f(x) = \frac{e^{\lambda x}}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b \frac{f(t)}{(x-t)^{\alpha+1}}e^{-\lambda t}dt.$$
Yang, Srivastava and Machado in 2015 presented the fractional derivative with the exponential function as
$$D_x^{(\alpha)}f(x) = \frac{(2-\alpha)\Im(\alpha)}{2(1-\alpha)}\frac{d}{dx}\int_a^x \exp\left(-\frac{\alpha}{1-\alpha}(x-t)\right)f(t)\,dt.$$
Atangana and Baleanu in 2016 recommended the fractional derivative with the Mittag- Leffler function as

$$D_x^{(\alpha)}\mathrm{f(x)} = \frac{\Im(\alpha)}{(1-\alpha)}\int_a^x E_\alpha\left(-\frac{\alpha}{1-\alpha}(x-t)^\alpha\right)\frac{df(t)}{dt}\mathrm{dt}$$

Sun, Hao, Zhang and Baleanu in 2017 reported the fractional derivative as

$$D_x^{(\alpha)}\mathrm{f(x)} = \frac{\Gamma(1+\alpha)}{(1-\alpha)^{1/\alpha}}\int_0^x exp\left\{-\frac{\alpha}{1-\alpha}(x-t)^\alpha\right\}\frac{df(t)}{dt}\mathrm{dt}.$$

Dehghan, Abbaszadeh and Deng in 2017 presented the fractional derivative as

$$D_+^{(\alpha)}\mathrm{f(x)} = \frac{1}{\Gamma(\alpha-1)}\int_0^x (x-t)^{\alpha+1}\ e^{-\lambda(x-t)}\frac{d^\alpha f(t)}{d|t|^\alpha}\ dt,$$

Where $\frac{d^\alpha f(t)}{d|t|^\alpha}$ is the Riesz fractional derivative.

In 2018, Sousa and de Oliverira introduced (initiated) the fractional derivative

$$D_+^{(\alpha)}\mathrm{f(x)} = \left(\frac{\varpi(x)}{\frac{d\varphi(x)}{dx}}\frac{d}{dx}\right)\int_a^x E_\alpha(\omega(t)(\varphi(x)-\varphi(t)^\alpha)\,\Omega(t)\frac{d\varphi(t)}{dt} \text{ where}$$

$$\varpi(\mathrm{x}) = \frac{M(\alpha(x))}{1-\alpha(x)},\ \omega(t) = -\frac{-\alpha(t)}{1-\alpha(t)}.$$

A distinguished tool for capturing the memory and hereditary properties of various materials and processes is the fractional derivative [Pathak, 2021]. This mathematical framework effectively models systems where the present state depends on past states, making it particularly useful in fields such as viscoelasticity, biology, and control theory. Historically, fractional calculus traces its roots to the Riemann-Liouville definition of the fractional integral of order α. This foundational concept laid the groundwork for modern developments in fractional derivatives, which now play a pivotal role in understanding complex, non-local behaviors that cannot be described using traditional integer-order calculus.

The Reimann- Liouville fractional derivative (RL) operators of order $\alpha > 0$ of a continuous function f: $(a, \infty) \to \mathbb{R}$ are defined as

$$C = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n\int_a^x (x-t)^{n-\alpha-1}\mathrm{f(t)dt} = \left(D^n\, I_{a^+}^{n-\alpha}\, f\right)(x)$$

And

$$(D_{b^-}^\alpha f)(x) = (-1)^n\left(\frac{d}{dx}\right)^n\int_x^b (t-x)^{n-\alpha-1}\mathrm{f(t)dt} = (-1)^n\left(D^n\, I_{a^+}^{n-\alpha}\, f\right)(x)$$

where n = $[x]$ +1

Integration by parts formula is given as

$$\int_a^b f(x)(\,^{C}D_{a^+}^\alpha g)\,(x)dx =$$

$$\int_a^b g(x)\,(D_{b^-}^\alpha\, f)(x)dx - \sum_{k=0}^{n-1}(-1)^{n-k}g^k\,(x)D^{n-k-1}\ I_{b^-}^{n-\alpha}\, f(x)|_a^b$$

When f(t) = t

$$D_{a^+}^\alpha\, f(t) = \frac{t^{1-\alpha}}{(\alpha-1)\Gamma(1-\alpha)}$$

Generalized Riemann- Liouville fractional derivative (GRLFD) or the Hilfer Fractional Derivative (HFD) of order n-1< $\alpha \le$ n and n ∈ N and type $0 \le \beta \le 1$ with respect to t is defined as

$$(D_{a^+}^{\alpha,\beta}\mathrm{y})(\mathrm{t}) = \left(I_{a^+}^{\beta(n-\alpha)}\right)\frac{d^n}{dt^n}\left(I_{a^+}^{(1-\beta)(n-\alpha)}\,y\right)(t).$$

Type β may support $D_{a^+}^{\alpha,\beta}$ for interpolating continuously between the classical Riemann-Liouville fractional derivative and the Caputo fractional derivative.

Applications as the result of development:

Fractional–order Speech Modelling: In discrete form, a speech signal can be expressed as a linear combination of its fractional derivatives as

$\hat{x}$ (n) = $\sum_{i=1}^{Q}\mu_k\, D^{\alpha_i}\, x$, where x(n) = n sample long speech analysis frame.

$\{\mu_k\}$ = the fractional LP coefficients

$D^\alpha x(t) = \lim_{h\to 0} h^{-\alpha}\sum_{j=0}^{\infty}\frac{(-\alpha)j}{j!}$ x(t-jh) ; h = the step size.

(2) Fractional RC Electrical Circuits:

The governing ODE of an RC electrical circuit is

$$\frac{dV(t)}{dt} + \frac{V(t)}{RC} = \frac{\mu(t)}{RC},$$

where V is the voltage, R = resistance, C = capacitance and (t) = source of volt.

(3) Modelling of children's physical Development:

D_x^α f(x) = $\frac{d^\alpha f(x)}{dx^\alpha} = \sum_{n=1}^{\infty} a_n\, n x^{n-1}$

It can give a better chance of predicting expected values of the child for future by using previous data obtained from the child development process.

(4) Fractional order mathematical modelling of COVID- 19:

$$D_t S(t) = \Delta - \lambda S - \frac{\alpha S(I+\beta A)}{N} - \Psi SQ$$
$$D_t E(t) = \frac{\alpha S(I+\beta A)}{N} + \Psi SQ - (1-\phi)\delta E - \phi\mu E - \lambda E$$
$$D_t I(t) = (1-\phi)\delta E - (\sigma+\lambda) I$$
$$D_t A(t) = \phi\mu E - (\rho+\lambda) A$$
$$D_t\, R(t) = \sigma I + \rho A - \lambda R$$
$$D_t Q(t) = \kappa I + \nu A - \varepsilon Q.$$

Here N = Total number of people.

Hidden dynamics of the infection in the mathematical models of the infectious disease is researched by the memory features of fractional derivative.

(5) Analyze Economic Growth Modelling:

Different fractional order models are introduced for accounting the nature of financial processes.

y(t) = $\sum_{k=1}^{9} C_k\, D^{\alpha_k} x_k(t)$, $C_k =$ constant weights, y = the GDP, x_k are the macro economic indicators variables.

(6) Application to Geo- Hydrology:

The Mittag-Leffler function incorporates the effect of memory, which is a fundamental principle for investigating groundwater flow. The memory effect is crucial for enabling water molecules to 'remember' their flow paths within a fractured network. Therefore, an accurate representation of the mathematical model requires considering this effect.

$S^{ABC} D_t^{\infty} H(r,t) = \frac{k}{t^{d-1-\theta}} \frac{\partial}{\partial r} (r^{d-1} \frac{\partial H(r,t)}{\partial r}$.

with H(r,0) = 0, $\lim_{t\to\infty} H(r,t) = 0$, d = dimension.

(7) Diffusion- Wave Equation of Groundwater Flow:

Common mathematical model of the radial groundwater flow to or from a well is

$$a^{ABC} D_t^{\alpha} H(r,t) + b^{ABC} D_t^{\alpha}\ \mathrm{H(r,t)} = \frac{D_t}{r} \frac{\partial}{\partial r}\left(r \frac{\partial H(r,t)}{\partial r}\right);$$

Here, 'a' and 'b' represent the ratio of immobile and mobile section porosities to the total porosity, and they satisfy the relation a + b = 1. The fractional derivative parameters for the mobile and immobile sections are defined as 0 < α ≤ 1 and 0 < β ≤ 1, respectively. D_f denotes the fractional diffusivity, and H represents the normalized groundwater depth.

(8) Sturm- Liouville BVP for integer order system:

These scenarios are commonly encountered in the analysis of problems related to simple harmonic motion, signal and image processing, heat conduction, string vibration, and wave propagation. As an example of heat equation such as

$\frac{\partial u}{\partial t}$ = c $\frac{\partial^2 u}{\partial x^2}$; u(x,0) = f(x), u(0,t) = 0, u(L,t) = 0.

(9) Applications in other areas:

The use of fractional mass-spring-damper systems, characterized by generalized fractional-order derivatives, has proven to be highly effective in capturing complex dynamics that traditional models struggle to address. These systems provide enhanced accuracy in modeling non-local and memory-dependent behaviors often encountered in real-world phenomena.

Remarkably, fractional derivative frameworks have found extensive applications across diverse fields, including biomedical engineering and underwater sediment analysis. In biomedical applications, fractional models have been employed to describe processes such as anomalous diffusion in tissues, arterial compliance, and biological signal processing, offering deeper insights into physiological dynamics. In underwater sediment analysis, fractional calculus aids in modeling fluid-particle interactions and the transport of sediments, accounting for long-range dependencies and complex flow behaviors that are difficult to capture using classical approaches. These advancements underscore the versatility of fractional calculus in enhancing the precision of mathematical models across multiple scientific domains. Fractional-order proportional-integral-derivative (PID) controllers have demonstrated exceptional efficacy in boosting crucial performance indicators of motion systems, including accuracy, bandwidth (velocity), and stability. Applying fractional calculus in image processing for edge detection can also improve detection criteria.

In an ever-evolving mathematical framework, a tapestry woven from differential equations emerges, where the enchanting concept of fractional calculus finds its rightful place. The curiosity of researchers has meandered through a myriad of disciplines, encompassing mathematics, physics, chemistry, engineering, finance, and the social sciences. This tome unfolds a contemporary narrative of fractional calculus and its realms of application, showcasing the profound impact of fractional calculus in unraveling challenges within biomedical engineering and applied sciences. It offers researchers, scientists, and clinicians a preparatory foundation for addressing new challenges involving fractional differential equations. The chapters are organized as follows:

i. New Directions in Fractional Differential Equations
ii. Proposals for Renewal in Fractional Behavior Studies
iii. On Riesz Derivative Problems
iv. Analysis of the Balance Equations and Corrections to Newton's Third Law
v. Modeling COVID-19 Pandemic Outbreak Using Fractional-Order Systems
vi. Damage and Fatigue Described by Wavelet Fractional Operators
vii. Fractional Calculus Applied to Large Processing
viii. Dynamics, Simulation, and Parameter Estimation of a Fractional Incommensurate Model Predicting COVID-19
ix. Fractional Calculus: A Reliable Tool for Solving Real-World Problems
x. Fractional Differential Equations and Their Applications in Circuit Theory
xi. Fractional Calculus: A Reliable Tool for Solving Real-World Problems
xii. Approximation of Mild Solutions of Semilinear Fractional Differential Equations
xiii. Fractional Calculus on Time Scales xiv. Extended Fractional Calculus

Recently, a surge of fascination has emerged surrounding fractional calculus, owing to its sophisticated applications across engineering, applied mathematics, finance, bioengineering, and various other domains. Outstanding investigations into fractional calculus were predominantly showcased in engineering publications during the latter part of the 20th century. This study is dedicated to presenting innovative and advanced applications of fractional calculus in various fields of science and engineering, inspiring further exploration of its profound significance. It highlights how fractional calculus provides powerful tools for modeling complex systems, capturing non-local dependencies, and addressing memory effects that traditional methods cannot fully represent.

1. **Applications to Transport in Fusion Plasma:** Fractional diffusion operators are used to develop phenomenological models for plasma transport, providing new insights into the behavior of fusion plasmas.
2. **Applications to Reaction-Diffusion Systems:** Fractional diffusion plays a significant role in reaction-diffusion systems, offering a deeper understanding of complex chemical and biological processes.
3. **Fractional-Order Model of Neurons in Biology:** The neurodynamics of the VOR are modeled using fractional calculus, explaining how eye rotations stabilize retinal images by counteracting head movements.
4. **Fractional Calculus in Electrochemistry and Fluid Flows:** Fractional-order models enhance the understanding of electrochemical processes and tracer fluid dynamics in complex systems.
5. **Electrical Circuits with Fractance:** In circuits with fractional-order impedance (fractance), traditional resistors and capacitors are replaced by components requiring fractional-order models for accurate analysis.
6. **Fractional-Order Dynamical Systems in Control Theory:** The exploration of time-domain characteristics in fractional-order systems is crucial for unraveling intricate challenges in the realm of control theory, presenting enhanced strategies for control.
7. **Generalized Voltage Divider:** Tree and chain fractance demonstrate the electrical behavior of non-integer-order impedance, along with conventional resistors and capacitors.
8. **Fractional Calculus in Robotics:** Fractional calculus enhances the precision of robotic motion and improves control methods, surpassing traditional control strategies.

9. **Applications in Analytical Science and Ultrasonic Wave Propagation:** Fractional calculus finds applications in ultrasonic wave propagation within human cancellous bone, as well as in analytical sciences.

10. **Fractional Calculus for Autonomous Vehicle Control:** Fractional-order controllers (FOC) are employed for lateral and longitudinal control, improving the path-tracking capabilities of autonomous electric vehicles.

Significance/Importance: The analysis of various methods and techniques in different fields, including science, mathematics, and areas like economics and agriculture, highlights the potential of fractional calculus.

Importance and Objectives: Understanding the methods and techniques applied across different fields, such as economics, agriculture, and business, will help advance the development of fractional calculus. This content will enhance readers' understanding and encourage researchers to further explore, extend, and apply established findings in their future studies. Collecting and analyzing data on published papers and new techniques will provide valuable insights for further development.

2. CONCLUSION

Fractional calculus has evolved through the revolutionary work of mathematicians like Leibniz, Bernoulli, Euler, Lagrange, Abel, Riemann, and many others. Today, fractional calculus is applied in advanced technologies, including robotics, image processing in biosciences, and addressing common issues such as milk and ghee adulteration and the detection of urea in milk. To date, approximately 100,000 papers and 100 books have been published on fractional differential equations. National, international, and regional researchers have worked collaboratively to advance fractional calculus.

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